

# Einstein's static universe

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## Abstract

Einstein's static model is the first relativistic cosmological model. The model is static, finite and of spherical spatial symmetry. I use the solution of Einstein's field equations in a homogeneous and isotropic universe — Friedmann's equation — to calculate the radius of curvature of the model (also known as *Einstein's universe*). Furthermore, I show, using a Newtonian analogy, the model's mostly known feature, namely, its instability under small perturbations on the state of equilibrium.

## 1 Introduction

In 1917, therefore, less than a hundred years ago, Albert Einstein (1879-1955) put forward the first relativistic cosmological model, i.e., a model based in the General Relativity Theory (GRT), that he had just finished ([1], chap. 8, [2], chap. 27, [3], chap. 14, [4], section 2).

The model, nowadays considered as surpassed, represented a most profitable seed of a series of theoretical studies which had the aim of understanding the general structure of the universe, both in space and time. Einstein's model is the starting point of relativistic cosmology. The model is static, with positive spatial curvature (closed), therefore, spatially bound — in other words, finite. It was static because this was the general view of the real universe at the time, and finite, because being so it avoided the necessity of infinite quantities as boundary conditions, an undesirable feature in any

physical theory. It is worthwhile mentioning that, in 1917, the hypothesis of a static universe was quite reasonable. The observations by Edwin Hubble (1889-1953), that would be consistent with non static solutions, had not yet been realized (see detailed discussion on this issue in [5]).

In order to achieve those characteristics it was necessary to counterbalance the attractive effects of gravity. Einstein introduced a constant in his field equations — the now famous *cosmological constant* —, as a repulsive term, at the right amount, to make possible the sort of solution he needed. Besides being in accordance with the common views of his epoch, of a static universe, Einstein aimed also to justify the ideas of the Austrian physicist and philosopher E. Mach (1838-1916) regarding the genesis of the property of inertia. According to Mach, the inertial mass of any body is due to the influence of the universe as a whole. Einstein agreed with such an idea and believed that his model connected local properties — the mass — with global properties — the cosmological constant ([3], p. 272). Incidentally, later on, Einstein's enthusiasm with respect to Mach's principle diminished and finally disappeared completely (see, for example, [6], p. 287).

Almost immediately after Einstein's proposition, the Dutch W. de Sitter (1872-1934), the Russian A. Friedmann (1888-1925) and the Belgian G. Lemaître (1894-1966) came up with alternative models to Einstein's static model, also based in GRT. The models of W. de Sitter, A. Friedmann and G. Lemaître have a peculiarity that does not exist in Einstein's model: they represent expanding universes. The light emitted by any galaxy arrives at an observer in a distant galaxy with its wavelength shifted towards the red, i.e., redshifted. In other words, light arrives with a wavelength larger than the wavelength at emission. Such a property does not exist in Einstein's model because it represents a static universe. W. de Sitter's model, on the other hand, has a feature that contributed to lessen its importance: it represents a universe completely without matter and radiation, where galaxies are interpreted as test particles immersed in the expanding space-time. It shares with Einstein's model the inclusion of a cosmological constant. As mentioned above, and as it will be explained in the next section, in Einstein's model the cosmological constant is responsible for the tendency to expansion that is exactly matched by the tendency to attraction due to matter and radiation. The latter do not exist in W. de Sitter's model, therefore, this model shows only expansion.

It was soon realized then that Einstein's model was unstable for small perturbations to the state of equilibrium. And finally, the British astrophysicist

Arthur Eddington (1882-1944) showed definitely that the model was unstable [7], lending capital doubts on its viability.

In the next section, I use Friedmann's equation, modified by the introduction of the cosmological constant, to calculate the radius of Einstein's universe. Next, I analyze the potential energy of a Newtonian analogy to show that this universe is in a state of unstable equilibrium. I finish with a discussion of Einstein's self-criticism about his first model of the universe.

## 2 The radius of Einstein's static universe

Friedmann's equation is a general solution of the field equations of GRT under the constraints of a homogeneous and isotropic fluid (see [8]). Einstein's field equations can be expressed in a synthetic form by means of the tensor formalism. Thus one has on the left side of the equation the energy-momentum tensor and on the right side the curvature tensor, which represents the system's space-time characteristics (see, for example, eq. 3.6 in [1]). In a simple manner, one can say that the mass and energy contents of the system *say* to space-time how to curve. Curved space-time *says* then to a test particle in it how to move.

The TRG field equations are, in fact, a system of non linear differential equations of extremely difficult solution. However, for a fluid that is homogeneous — same density everywhere — and isotropic — same properties at all directions —, as mentioned above, the system of equations is simplified allowing for analytical solutions, such as, for example, Friedmann's equation.

Friedmann's equation has on the left-hand side the energy terms and on the right side the curvature term. It is written, in terms of the curvature constant of the system,  $K_o$ , as ([1], eq. 2.19):

$$\left(\frac{dR}{dt}\right)^2 - \frac{8\pi G}{3}\rho R^2 = -K_o c^2, \quad (1)$$

where  $R$  is the scale factor and  $\rho$  is the total density in  $R(t)$ .  $G$  is the universal gravitational constant and  $c$  is the speed of light in vacuum. The density  $\rho$  varies with time and its present observed value is approximately  $10^{-30}$  g/cm<sup>3</sup>. The curvature constant is, for a closed spherical universe,  $K_o = +1/\mathcal{R}^2$ , and  $\mathcal{R}$  is the radius of curvature of the spherical space. For a critical (or flat) model  $\mathcal{R} \rightarrow \infty$ , and, therefore,  $K_o = 0$ . The open universe has an imaginary

radius of curvature, meaning that it has a negative constant of curvature  $K_{\circ} = -1/\mathcal{R}^2$  (hyperbolic space).

This equation was obtained, for the first time, by the Russian Alexander Friedmann in 1922. It is used here in the discussion of Einstein's model because it makes much more simpler the derivation of both the radius of the universe and the investigation of the model's stability. Historically, though, that was not the way followed by Einstein, because his model was devised in 1917.

Eq. 1 can be modified, without violating GRT, by adding a constant, conveniently expressed as  $1/3\Lambda c^2$ , on the left-hand side of the equation. This additional term can also be considered as a density term  $\rho_{\Lambda} = \Lambda c^2/8\pi G$ . Hence, one has

$$\left(\frac{dR}{dt}\right)^2 - \left(\frac{8\pi G}{3}\rho + \frac{1}{3}\Lambda c^2\right) R^2 = -K_{\circ}c^2, \quad \text{or} \quad (2)$$

$$\left(\frac{dR}{dt}\right)^2 - \frac{8\pi G}{3}(\rho + \rho_{\Lambda}) R^2 = -K_{\circ}c^2, \quad \text{and finally} \quad (3)$$

$$\left(\frac{dR}{dt}\right)^2 - \frac{8\pi G}{3}\frac{\rho_{\circ}}{R} - \frac{1}{3}\Lambda c^2 R^2 = -K_{\circ}c^2, \quad (4)$$

with  $\rho(t)R(t)^3 = \rho(t_{\circ})R(t_{\circ})^3$ , or,  $\rho R^3 = \rho_{\circ}$ , where  $t_{\circ}$  is the present time,  $\rho_{\circ}$  is the density in  $t_{\circ}$  and  $R(t_{\circ})$  is, conventionally, set to 1. The transformation  $\rho R^3 = \rho_{\circ}$  is nothing more than the expression of mass conservation in an evolving universe (density times volume, i.e., mass, is constant), which is valid also, of course, for the special case of a static universe.

The cosmological constant  $\Lambda$  has the physical dimension of  $1/\text{length}^2$ . According to the cosmologist Wolfgang Rindler ([9], p. 303), “*The  $\Lambda$  term [...] seems to be here to stay; it belongs to the field equations much as an additive constant belongs to an indefinite integral*”. While, mathematically, the cosmological constant preserves the validity of GRT's field equations, physically, it leads to multiple possible consequences in the behavior of model universes.

Differentiating eq. 4 with respect to time, yields:

$$2\dot{R}\ddot{R} + \frac{8\pi G}{3}\frac{\rho_{\circ}}{R^2}\dot{R} - \frac{2}{3}\Lambda c^2 R\dot{R} = 0, \quad (5)$$

which can be simplified to

$$\ddot{R} + \frac{4\pi G}{3} \frac{\rho_o}{R^2} - \frac{1}{3} \Lambda c^2 R = 0. \quad (6)$$

With constant  $R$ , eq. 6 clearly shows that  $\Lambda$  can be fine-tuned to yield  $\ddot{R} = 0$ , thus, implying a static solution, which was precisely Einstein's desire.

As mentioned above, in Einstein's static model the constant of curvature is  $K_o = 1/\mathcal{R}^2$ . With this and making the scale factor  $R \equiv R_E = 1$  in eqs. 4 and 6, one gets the following two relations:

$$-\frac{8\pi G}{3} \rho_o - \frac{1}{3} \Lambda c^2 = -\frac{c^2}{\mathcal{R}^2} \quad (7)$$

$$\frac{4\pi G}{3} \rho_o - \frac{1}{3} \Lambda c^2 = 0. \quad (8)$$

Inserting eq. 8 in eq. 7 gives

$$4\pi G \rho_o = \frac{c^2}{\mathcal{R}^2} \quad (9)$$

or

$$\mathcal{R} = \frac{c}{\sqrt{4\pi G \rho_o}}, \quad (10)$$

that is the radius of curvature of Einstein's static universe.

What is its numerical value? For the sake of illustration, let us take  $\rho_o = 3H_o^2/8\pi G$ , namely, the density of Friedmann's critical model, also known as the Einstein-de Sitter model. Here,  $H_o$  is Hubble's constant (see [1] and [2] for more details about such a model). Then, one gets  $\mathcal{R} = \sqrt{2/3}(c/H_o) = 3.4 \text{ Gpc} = 11 \text{ Gly}$ , with  $H_o = 72 \text{ km s}^{-1}\text{Mpc}$  (cf. [10]).

It is worthwhile stressing that the above calculation of  $\mathcal{R}$  is just illustrative, having no real physical meaning. In the days when Einstein put forward his static model, the value used for the density was the observed density, which, coincidentally, in order of magnitude, did not differ from the value exemplified above.

### 3 Study of stability

As shown in the beginning of the preceding section, Friedmann's equation (eq. 1) has the energy terms on its left-hand side and the curvature term —

which is constant — on its right side. A Newtonian analogy may be built from eq. 4. Such an equation represents the conservation of total energy, applied to the cosmic fluid. We shall use, for the analogy, Friedmann’s equation modified with the addition of the cosmological constant, in the form of eq. 4.

The right-hand side term represents the total energy of the system — negative, i.e., a bound system, as is the case in Einstein’s model. The first term on the left-hand side represents the kinetic energy of the cosmic fluid element, the second term its gravitational potential energy and the third term — of the  $-1/2kx^2$  kind — represents a sort of repulsive “elastic” cosmic potential energy. This last term, in Friedmann’s equation, could be thought of as an intrinsic stress in the space-time tissue, quantified by the cosmological constant. In the analogous Newtonian construction, it is regarded as an elastic potential energy of a string, with the important difference of being a *negative* energy term. The second term will be, then, represented by  $U_G = -1/R$  and the third one by  $U_\Lambda = -1/2R^2$ .

The radial forces related to these potential energies can be calculated by  $F = -dU/dR$ , yielding  $F_G = -1/R^2$  and  $F_\Lambda = +R$ , the first, an attractive force — driven by gravitation — and the second one, a repulsive force — driven by the cosmic “elasticity”, much like the same as a rubber sheet would do — the space-time tissue — upon a body that rests on it. These two forces balance exactly in Einstein’s static universe.

Therefore, the conservation of energy in the Newtonian analogy may be written as

$$\frac{1}{2}mv^2 + U_G + U_\Lambda = -E \quad (11)$$

$$\frac{1}{2}mv^2 - \frac{1}{R} - \frac{1}{2}R^2 = -E, \quad (12)$$

where  $-E < 0$  is the system’s total energy. Fig. 1 shows the total potential energy function  $U = U_G + U_\Lambda$ . It is quite apparent that the point of equilibrium —  $F = -dU/dR = 0$  — represents an unstable equilibrium. Precisely what we would like to show.

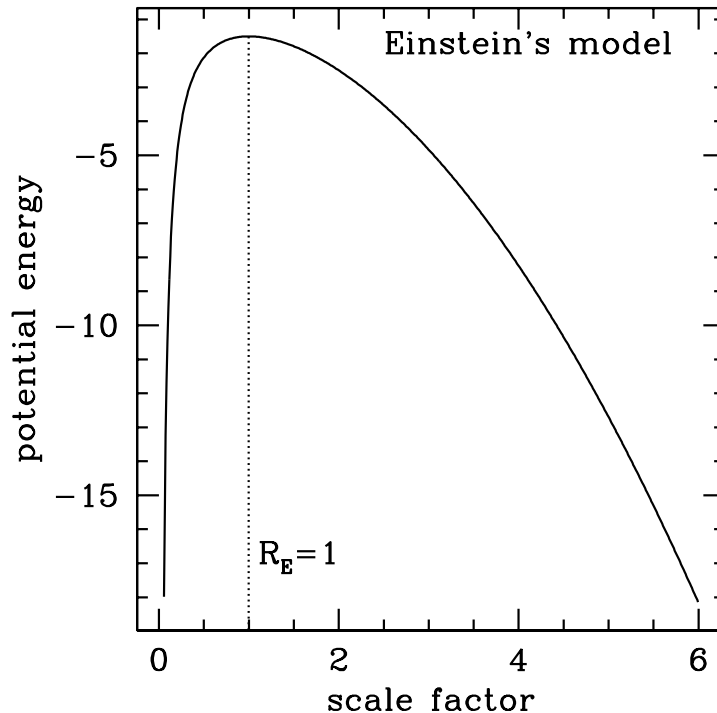


Figure 1:  $\Lambda$ -shaped diagram: the potential energy — in arbitrary units — for the Newtonian analogy of Einstein’s static model. Notice that the equilibrium at  $R = R_E$  is an unstable one. Any small perturbation at  $R_E$  makes either the universe to collapse or diverge to  $R \rightarrow \infty$ .

## 4 Final remarks

Soon after Einstein put forward his cosmological model, two almost simultaneous events, in the beginning of the 1920s, changed in a dramatic way the scientific view of the universe. One of them was the discovery of the systematics exhibited by the spectral shifts of the radiation emitted by extragalactic nebulae, undertaken by Edwin Hubble. The other one was the discovery of new solutions of Einstein’s field equations, by Friedmann (see eq. 1, above, used in section 2), that implied in dynamical models. The universe could be either in expansion or in contraction, and the first possibility was consistent with Hubble’s observations. There was not anymore the necessity of a static model.

It is rather well known Einstein's reaction to these great events.

The renowned theoretical physicist John Archibald Wheeler (1911-2008) tell us that, once, as a young scientist, he went along with Einstein and George Gamow (1904-1968), in the Institute of Advanced Studies, in Princeton, when he heard Einstein confess to Gamow that the cosmological constant had been “*the biggest blunder of my life*” (cf. [11], p. G-11).

Obviously, Einstein was not a fool, and the inclusion of  $\Lambda$  in his field equations, definitely, was not a blunder at all. It increased in a substantial way the applicability of GRT, without causing damages from the formal point of view, as mentioned in section 2.

In fact — and it is something that probably Einstein did not want to recognize —, his real big blunder was to put forward a model that was clearly unstable. The fact that he was not aware of that is that causes a big surprise. As we saw, in section 3, a simple analogous in classical reasoning makes clear such a very serious failure.

## References

- [1] R.E. de Souza, *Introdução à Cosmologia*, (EDUSP, São Paulo, 2004).
- [2] B.W. Carroll, D.A. Ostlie, *An Introduction to Modern Astrophysics*, (Addison-Wesley Publ. Co., Inc., Reading, 1996).
- [3] E. Harrison, *Cosmology – The Science of the Universe*, (Cambridge University Press, Cambridge, 2000).
- [4] I. Waga, *Cem anos de descobertas em cosmologia e novos desafios para o Século XXI*, 2005, RBEF **27**, pp. 157-173,  
[http://www.sbfisica.org.br/rbef/pdf/v22\\_163.pdf](http://www.sbfisica.org.br/rbef/pdf/v22_163.pdf).
- [5] D. Soares, *O efeito Hubble*, 2009,  
<http://www.fisica.ufmg.br/~dsoares/ensino/efhub.pdf>.
- [6] A. Pais, *Subtle is the Lord: The Science and the Life of Albert Einstein*, (Oxford University Press, Oxford, 1982).
- [7] A.S. Eddington, *On the stability of Einstein's spherical world*, 1930, MNRAS **90**, pp. 668-678



- [8] A. Viglioni, D. Soares, *Note on the classical solutions of Friedmann's equation*, 2010, <http://arxiv.org/abs/1007.0598>.
- [9] W. Rindler, *Relativity – Special, General, and Cosmological*, (Cambridge University Press, Cambridge, 2006).
- [10] W.L. Freedman et al., *Final Results from the Hubble Space Telescope Key Project to Measure the Hubble Constant*, 2001, ApJ **553**, pp. 47-72.
- [11] E.F. Taylor, J.A. Wheeler, *Exploring Black Holes, Introduction to General Relativity*, (Addison Wesley Longman, San Francisco, 2000).